CAT 2

Homotopy

Last Time...

- We described the "Gromov-Hausdorff distance" between (pairs of) finite metric spaces
- Defined "simplicial complexes", which link combinatorics to piecewise-linear geometry via the assignment $K \to |K|$
- Constructed Vietoris-Rips and Čech complexes from metric spaces at various scales.

Goal 0 To better understand the structure of simplicial complexes. Answer when two simplicial complexes are "similar".

Def 1 a) Two continuous functions $f_0, f_1 : X \to Y$ between topological spaces X and Y are said to be HOMOTOPIC (to each other) if there exists a third function

$$\boxed{\Theta : X \times [0,1] \to Y}$$

, also continuous,

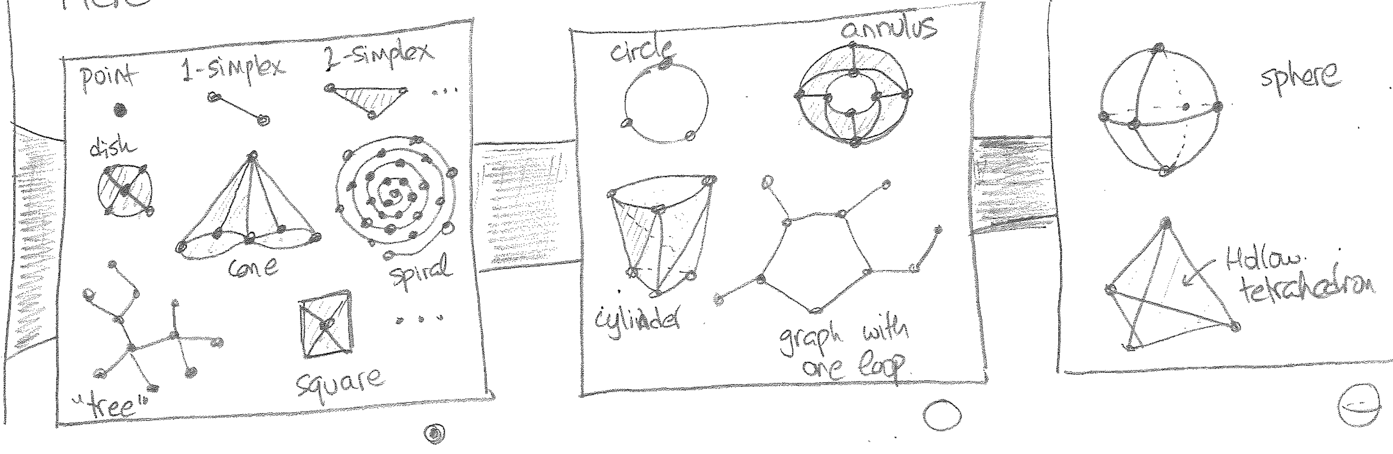
so that
- $\Theta(x,0) = f_0(x) \quad \forall x \in X$, and
- $\Theta(x,1) = f_1(x) \quad \forall x \in X$.

H is a "path in the space of all functions $X \to Y$"

b) Two topological spaces X and Y are called HOMOTOPY EQUIVALENT if there are continuous functions $f : X \to Y$ and $g : Y \to X$ so that the composite $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X

"Identity" $\mathrm{id}_X : X \to X$ sends each point to itself

Eg 2 Here are "some examples" of homotopy-equivalent spaces



Note 3 We say two simplicial complexes K and L are homotopy equivalent whenever their geometric realizations $|K|$ and $|L|$ are homotopy-equivalent.

Things to check... 4.

a) Homotopy is an equivalence relation on functions.

b) Homotopy equivalence is an equivalence relation on spaces.

c) There is no homotopy between • and • •

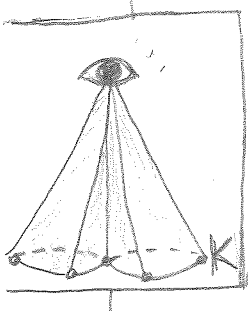
d) Homeomorphic spaces are homotopy equivalent

Def 4 A space is called <u>CONTRACTIBLE</u> if it is homotopy equivalent to a <u>point</u>. Examples of such include

a) every tree (= connected graph with no cycles)

b) every simplex (we already checked •——•)

c) every "cone". A <u>CONE</u> on a simplicial complex K is a new simplicial complex defined as follows:



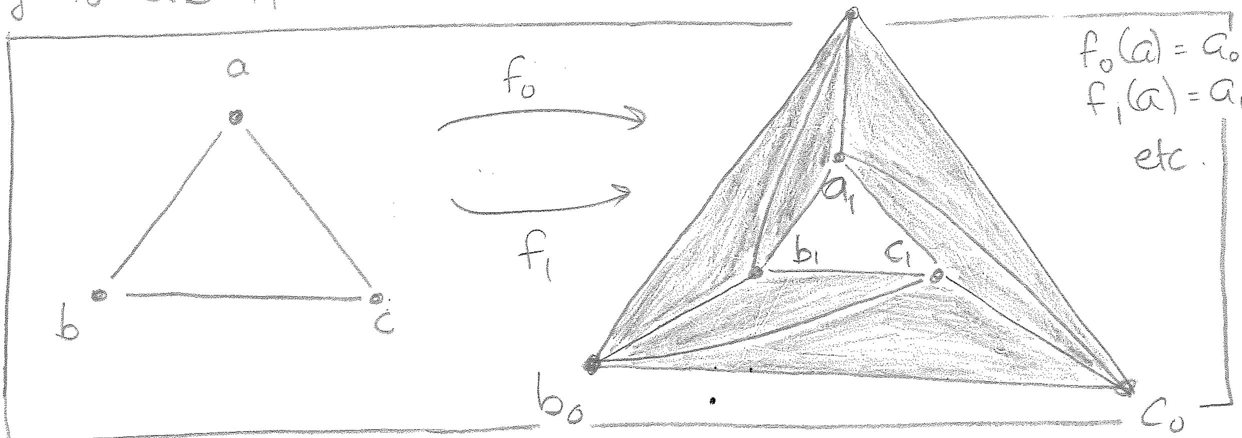
- Its vertices are $K_0 \cup$ "◉" ← $\boxed{\text{new vertex!}}$
- Its i-simplices are
  - all i-simplices of K, PLUS
  - $\alpha \cup \{◉\}$ for each $(i-1)$-simplex α of K.

Q5 How can we show two spaces to be homotopy equivalent? The obvious sub-question is: how can we show that two maps are homotopic?

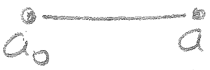
Remarkably, the <u>key</u> to answering both questions is to build as large a database of contractible spaces as possible!

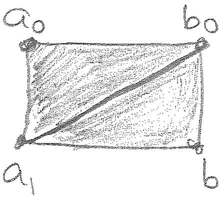
Thm 6 **"Carrier Lemma"** Let $f, g : K \to L$ be two simplicial maps so that $\forall\, \alpha \in K$, there exists a contractible subcomplex $L_\alpha \subseteq L$ satisfying $f(\alpha) \in L_\alpha$ and $g(\alpha) \in L_\alpha$. Then f and g are homotopic.

Eg 7 Consider two embeddings of a circle into an annulus; say f_0 and f_1:



In order to use Thm 6 for showing f_0 and f_1 are homotopic, we'd build subcomplexes

$L_a =$  (similarly b & c)

$L_{ab} =$  (similarly ac, bc)

+ <u>all</u> of these $L_\bullet$'s should be contractible!

Prop 8 Let K be the simplicial complex given by a single d-dimensional simplex $(d \geq 0)$ and all of its faces. Then $|K|$ is contractible.

Pf Let $\{v_0, \dots, v_d\} \in \mathbb{R}^n$ be affinely independent so that $|K|$ is the subset $\{\sum_{i=0}^{d} \lambda_i v_i \mid \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1\}$.

Consider the map

$$\boxed{|K| \times [0,1] \xrightarrow{\theta} |K|}$$

which sends the pair $(\sum_{i=0}^{d} \lambda_i v_i, t)$ to the point
$\left[(\lambda_0 + (1-t)(\lambda_1 + \dots + \lambda_d)) \cdot v_0 + t \cdot \sum_{i=1}^{d} \lambda_i \cdot v_i\right]$. Check that it homotopy from the identity map $(t=1)$ to the map $K \to \{v_0\}$ (at $t=0$).

Cor 9 Let $f, g : K \to L$ be simplicial maps so that $\forall\, \sigma \in K$ the simplices $f(\sigma)$ and $g(\sigma)$ have a common co-face, i.e., $\exists\, \tau \in L$ so that

$$f(\sigma) \leq \tau \geq g(\sigma).$$

Then f, g are homotopic!

Pf Carrier lemma + previous proposition.

NERVES

Def 10 Let X be any topological space and let $\{\mathcal{U}_i\}_{i=1}^{n}$ be a cover of X by open subsets (not empty!)

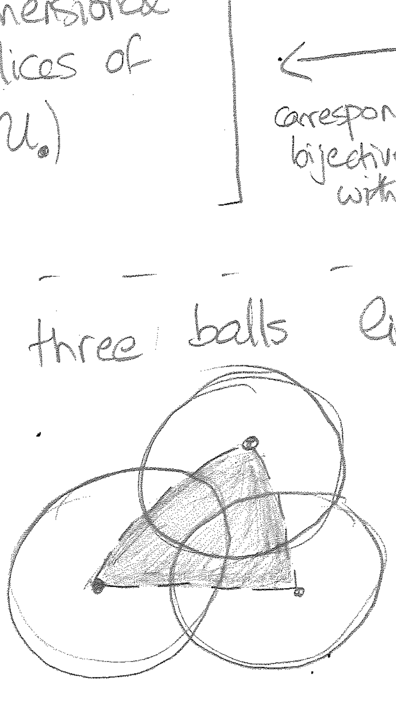
In other words: $\mathcal{U}_i \subseteq X$ open $\forall i$ and $X = \bigcup_{i=1}^{n} \mathcal{U}_i$

The NERVE of $\{\mathcal{U}_\bullet\}$ is a simplicial complex, defined as follows:

[d-dimensional simplices of $N(\mathcal{U}_\bullet)$] $\longleftrightarrow$ (correspond bijectively with...) [$(d+1)$-fold non-empty intersections of sub-sets of $\{\mathcal{U}_i\}$]

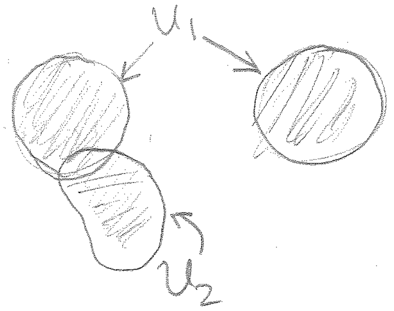
Eg 11 Union of three balls like this

a)

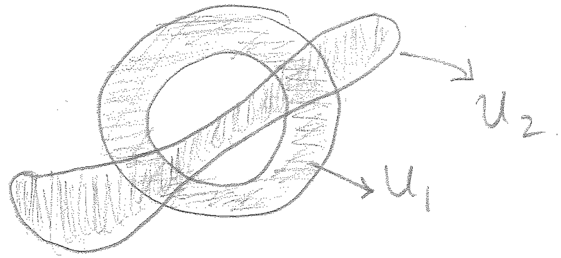


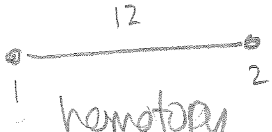
Each ball gets a vertex, each pair-wise intersection an edge, each triple intersection a 2-simplex... and so forth!

b) But we can also have:



or



Nerves are = 

for both, but these two are NOT homotopy equivalent.

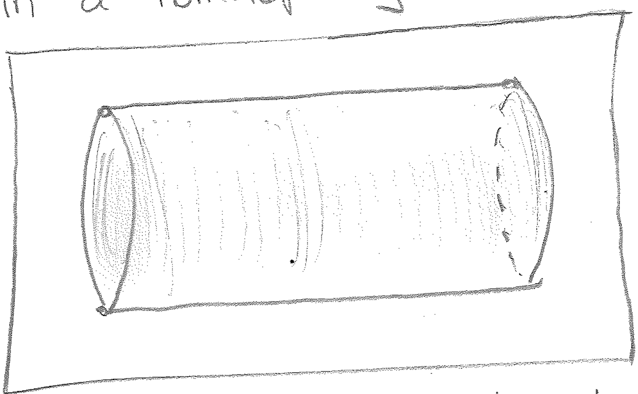
Thm 12 (Nerve Thm)

Let $\{\mathcal{U}_\bullet\}$ be an open cover of a topological space $\underline{X}$ with the following property. Given any subcollection $\mathcal{U}_{i_0}, \ldots, \mathcal{U}_{i_k}$ of $\{\mathcal{U}_\bullet\}$, the intersection $\bigcap_{j=0}^{k} \mathcal{U}_{i_j}$ is either empty or contractible. Then, the Nerve $N(\mathcal{U}_\bullet)$ is homotopy-equivalent to $\underline{X}$.

Such covers are called "good covers"

Note 13

This result lets us "triangulate" many interesting spaces in a homotopically faithful way. Try a cylinder:



[Note: circles are not contractible]

Find a cover with contractible intersections, what is its nerve?

Cor 14

Let $P \subseteq \mathbb{R}^n$ be a finite point set and let $P^{+\varepsilon}$ be the union of ε-balls around P. Then, for each $\varepsilon \geq 0$, the Čech complex $\check{C}_\varepsilon(P)$ is homotopy-equivalent to $P^{+\varepsilon}$. This is NOT true if you replace $\check{C}_\varepsilon$ with VR_ε...

Note 15

Homotopy, in spite of its conceptual appeal, is not easy to "compute". We will try to find more computable invariants, i.e., properties of simplicial complexes that do not change under homotopic transformations. Before we leave this world behind, a word about my FAVOURITE THEOREM, or at least a special case thereof.

Thm 16

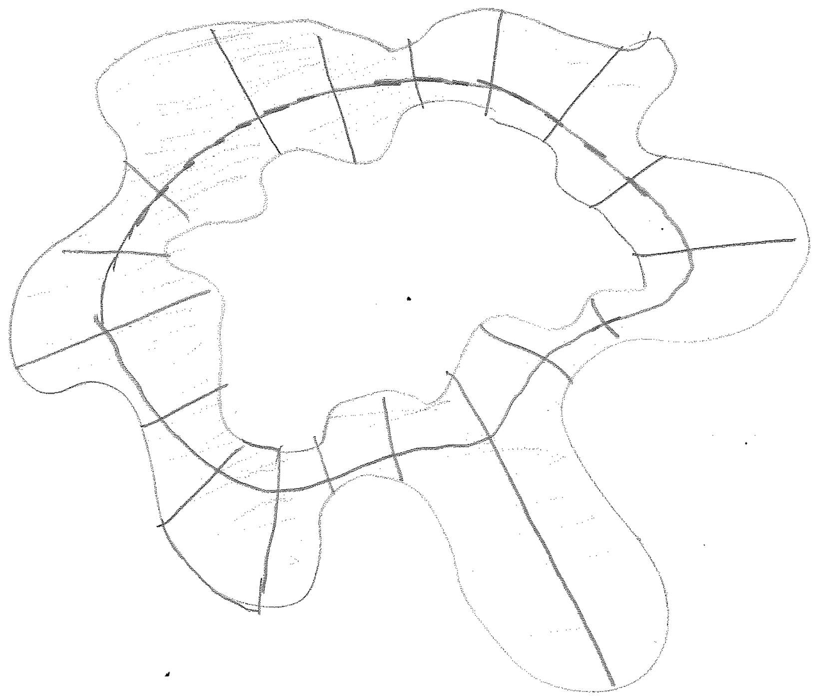
Quillen's Thm A for simplicial complexes

Let $f: K \to L$ be a simplicial map with the property that $\forall \tau \in L$, the "inverse image"

$$\{\sigma \in K \mid f(\sigma) \leq \tau\}$$

is contractible. Then f is a homotopy equivalence.

You can use this Theorem to do MAGIC.



Eg K = weird annulus, L = embedded loop. The fibers are all lines ($\sim$ 1 simplices) or solid rectangle-ish regions ($\sim$ 2 simplices), hence contractible!